Heterogeneous Spectrum Sharing with Rate Demands in Cognitive MIMO Networks

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***Abstract*—We are interested in addressing a fundamental question: what are conditions under which an ad hoc cognitive radio MIMO (CMIMO) network can support a given rate-demand profile, defined as the set of rates requested by individual links? From an information theoretic view, a rate profile can be supported if it is within the network capacity region. However, the network capacity region of interfering MIMO networks is essentially unknown. In dynamic spectrum access, the problem is even more challenging due to the dynamics of primary/legacy users (PUs), resource constraints, and the heterogeneity of opportunistic spectrum (i.e., the set of available channels varies from one to another). Considering a non-centralized setup, we address the above question in a noncooperative game framework where each CMIMO link independently optimizes its spectrum, power allocation, and MIMO precoders to meet its rate demand. We derive sufficient conditions for the existence of a NE are derived. These conditions establish an explicit relationship between the rate-demand profile and interference from PUs, CMIMO network's interference, and CMIMO nodes' power budget. We also show that a NE, if exists, is unique. Our results help to characterize the network capacity region of CMIMO networks.**

***Index Terms*—Cognitive radio, MIMO, Nash equilibrium, noncooperative game, rate demand, interfering network capacity.**

I. Introduction

Consider an interfering CR MIMO (CMIMO) network in which each link wishes to minimize its transmit power while meeting a given rate demand. The problem can be modeled as a noncooperative game, referred to as *power minimization* (PM) game. Such a PM game is different from the *rate maximization* (RM) game (e.g., [1] [2] [3]), in which nodes individually maximize their rates. Whereas the players' strategic spaces in a RM game are independent, these strategic spaces exhibit complex coupling in a PM game. This is because the strategic space of a link in the RM game is defined by its available resources, e.g., power, available channels, antennas, etc., which do not depend on other players' actions. In contrast, in a PM game with rate constraints, the strategic space of a player is not only shaped by its resources but also its achievable rate (to meet the rate demand), which is a function of other players' actions. The interdependence of the strategic spaces makes the analysis of PM games much more challenging than RM games.

As an example, it can be proved that the RM game always admits a NE [2]. By contrast, the PM game may not have a NE (e.g., the rate demands are beyond the network capacity region). Moreover, under resource constraints (e.g., power budgets), the strategic space of a player under the PM game can be empty (e.g., when the power budget is not sufficient to support the rate demand given interference from other transmitters). In the context of a CR network, the dynamics of primary users (PUs) also affect if a requested CR rate can be met or not. It is this possible emptiness of strategic spaces that prevents us from directly applying techniques used to study the RM games of MIMO systems to our setup. The projection method (onto a nonempty compact and convex space) in the context of fixed point theory [4] and the variational inequality theory [3] [5] have been instrumental in tackling the RM game. However, these techniques require nonempty strategic spaces.

Another challenge is the spectrum heterogeneity of CR communications. Due to the spatiotemporal variations of spectrum opportunities, a channel that is temporarily available for one CR user may not be available to other CR users. This leads to a CR network with *heterogeneous spectrum sharing*. Traditional RM and PM games [1] [2] [3] [6] [7] often assume *homogeneous spectrum sharing* setting in which the set of idle channels are the same at all nodes.

The goal of this paper is to investigate the conditions under which a given rate profile can be supported by an interfering CMIMO network with heterogeneous spectrum sharing. Using game theory, recession analysis, and variational inequality theory, we derive sufficient conditions that guarantee a given rate profile. Intuitively, these conditions are met if the CRs' power budget is sufficient enough to satisfy the rate demands, the requested rates are not too high to harm PU receptions, the PUs' interference to CRs is not too strong, and the CR interference is not too severe. The four conditions are quantified in a way that allows a node to decide its appropriate rate. We also show that if a NE exists for the underlying PM game, it must be unique. Interestingly, by removing resource constraints and set the number of antenna to 1, our sufficient conditions become necessary and reduce to those in [8] [9].

Throughout the paper, $(.)^*$ denotes the conjugate of a matrix, $(.)^H$ denotes its Hermitian transpose, tr$(.)$ denotes its trace, $|.|$ denotes its determinant, $||.||$ denotes the Euclidean (or Frobenius) norm, and $(.)^T$ denotes the transpose. $\mathrm{eig}_{\max}(.)$, $\mathrm{eig}_{\min}(.)$, and $\mathrm{diag}_s(.)$ indicate the maximum, minimum eigenvalue, and the diagonal element (s,s) of a matrix, respectively. Matrices and vectors are bold-faced.

II. Problem Statement

A. System Model

Consider a multi-channel CMIMO network that coexists with several PU networks in a rich-scattering environment (to facilitate MIMO spatial multiplexing). The network consists of N transmitter-receiver pairs (links), denoted by $\Phi_N \stackrel{\text{def}}{=} \{1, 2, \ldots, N\}$. Each CR node is equipped with M antennas. The set of temporarily idle channels at link i is denoted by $\mathbf{S}_i$. In general, $\mathbf{S}_i \neq \mathbf{S}_j$ for two links i and j. The network's opportunistic spectrum is the union of available-channel sets from all links, consisting of K orthogonal (not necessarily contiguous) channels with central frequencies f_1, f_2, …, f_K, denoted by $\Psi_K \stackrel{\text{def}}{=} \{1, 2, \ldots, K\} = \bigcup_{i=1}^{N} \mathbf{S}_i$. Each CR

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i can simultaneously communicate over multiple frequencies (e.g., using non-contiguous OFDM).

The transmitter of a CR link can send up to M independent data streams over each channel. Let $\mathbf{x}_u^{(k)}$ be an $M \times 1$ column vector, consisting of M information symbols (from M data streams), sent on link u using the channel with central frequency f_k (hereon also referred to as channel f_k for short). The radiation pattern and power allocation for the M streams of link u on channel f_k are determined by its precoding matrix $\tilde{\mathbf{T}}_u^{(k)}$. The actual transmit vector on channel f_k at the radio interface is $\tilde{\mathbf{T}}_u^{(k)}\mathbf{x}_u^{(k)}$. We allow for spectrum sharing among various CR links. On channel f_k, the signal vector $\mathbf{y}_u^{(k)}$ at the receiver of link u is given by:

$$\mathbf{y}_u^{(k)} = \mathbf{H}_{u,u}^{(k)}\tilde{\mathbf{T}}_u^{(k)}\mathbf{x}_u^{(k)} + \sum_{j\in\Phi_N\setminus\{u\}} \mathbf{H}_{u,j}^{(k)}\tilde{\mathbf{T}}_j^{(k)}\mathbf{x}_j^{(k)} + \mathbf{N}_k \quad (1)$$

where $\mathbf{H}_{u,u}^{(k)}$ is an $M \times M$ channel gain matrix on channel f_k of link u. Each element of $\mathbf{H}_{u,u}^{(k)}$ is a multiplication of a distance- and channel-dependent attenuation term, and a complex Gaussian variable (with zero mean and unit variance) that reflects multi-path fading. $\mathbf{H}_{u,j}^{(k)}$ denotes the cross-channel gain matrix from the transmitter of link j to the unintended receiver of link u, $u \neq j$. The second term in (1) represents interference from transmitters of CR links $j \neq u$ that share channel f_k with link u. $\mathbf{N}_k$ is an $M \times 1$ complex Gaussian noise vector with covariance matrix $\mathbf{I}_k = (1 + I_{pu}(k))\mathbf{I}$, representing the floor noise with unit variance plus (whitened) interference $I_{pu}(k)$ from PUs on channel f_k.

We assume that interference cancellation is not used. A receiver decodes its data streams by treating interference from other transmitters as colored noise. The Shannon rate over link u on channel f_k is [10]:

$$R_u^{(k)} = \log |\mathbf{I} + \tilde{\mathbf{T}}_u^{(k)H}\mathbf{H}_{u,u}^{(k)H}{\mathbf{C}_u^{(k)}}^{-1}\mathbf{H}_{u,u}^{(k)}\tilde{\mathbf{T}}_u^{(k)}| \quad (2)$$

where $\mathbf{C}_u^{(k)}$ is the noise-plus-interference covariance matrix at the receiver of link u over channel f_k:

$$\mathbf{C}_u^{(k)} = (1 + I_{pu}(k))\mathbf{I} + \sum_{j\in\Phi_N\setminus\{u\}} \mathbf{H}_{u,j}^{(k)}\tilde{\mathbf{T}}_j^{(k)}\tilde{\mathbf{T}}_j^{(k)H}\mathbf{H}_{u,j}^{(k)H}.$$

The total channel rate over all frequencies of link u is:

$$R_u = \sum_{k\in\mathbf{S}_i} R_u^{(k)}. \quad (3)$$

PU protection is provided in the form of database-authorized access and frequency-dependent power masks on CR transmit powers. Note that the FCC [11] recently imposed power masks even for idle channels, if such channels are adjacent to PU-occupied channels. Let $\mathbf{P}_{mask} \stackrel{\text{def}}{=} (P_{\text{mask}}(f_1), P_{\text{mask}}(f_2), \ldots, P_{\text{mask}}(f_K))$ denote the power mask vector. We require:

$$\sum_{s=1}^{M} P_{s,k}^{(u)} = \text{tr}(\tilde{\mathbf{T}}_u^{(k)}\tilde{\mathbf{T}}_u^{(k)H}) \leq P_{\text{mask}}(f_k) \quad (4)$$

where $P_{s,k}^{(u)}$ denotes the power allocated on channel f_k (frequency dimension) over antenna s (space dimension) for the transmitter of link u. If channel f_k is not available for link u, $P_{s,k}^{(u)} = 0, \forall s = 1 \ldots M$.

Due to spectrum heterogeneity, we require link u not to transmit on channels that are not in $\mathbf{S}_u$ by imposing a link-dependent power-mask vector $\mathbf{P}_{\text{mask}}(u)$. For link u, $\mathbf{P}_{\text{mask}}(u) \stackrel{\text{def}}{=} (P_{\text{mask}}(u, f_1), P_{\text{mask}}(u, f_2), \ldots, P_{\text{mask}}(u, f_K))$, where $P_{\text{mask}}(u, f_k) = 0$ if $f_k \notin \mathbf{S}_u$, and $P_{\text{mask}}(u, f_k) = P_{\text{mask}}(f_k)$ otherwise. Note that $\mathbf{P}_{\text{mask}}(u)$ differs from one link to another. We impose following constraints:

$$\begin{aligned} &\text{C1: } c_u \leq R_u, \ \ \forall u \in \Phi_N \\ &\text{C2: } \text{tr}(\tilde{\mathbf{T}}_u^{(k)}\tilde{\mathbf{T}}_u^{(k)H}) \leq P_{\text{mask}}(u, f_k), \ \ \forall k \in \Psi_K, \forall u \in \Phi_N \\ &\text{C3: } \sum_{k\in\Psi_K} \text{tr}(\tilde{\mathbf{T}}_u^{(k)}\tilde{\mathbf{T}}_u^{(k)H}) \leq P_{\max}, \ \ \forall u \in \Phi_N. \end{aligned} \quad (5)$$

where C1 ensures that all links achieve their rate demands, C2 ensures that the frequency-dependent power masks are satisfied, and C3 presents a maximum-power budget constraint ($P_{\max}$) at node u (we assume nodes have an identical power budget).

B. Noncooperative Game Formulation

Each CR link represents a player in the PM game who aims at maximizing its utility, defined as the negative of its transmit power. The game's strategic space Q is the union of the strategic spaces of various players, subject to constraints C1, C2, C3 in (5). Each player u competes against others by selecting his strategic action of K precoders, denoted by $\tilde{\mathbf{T}}_u \stackrel{\text{def}}{=} (\tilde{\mathbf{T}}_u^{(1)}, \tilde{\mathbf{T}}_u^{(2)}, \ldots, \tilde{\mathbf{T}}_u^{(K)})$. $\tilde{\mathbf{T}}_u$ is an $M \times KM$ block matrix, comprised of K $M \times M$ matrices. The payoff for player u, given below, is a function of its action $\tilde{\mathbf{T}}_u$ as well as other players' actions, $\tilde{\mathbf{T}}_{-u} \stackrel{\text{def}}{=} (\tilde{\mathbf{T}}_1, \tilde{\mathbf{T}}_2, \ldots, \tilde{\mathbf{T}}_{u-1}, \tilde{\mathbf{T}}_{u+1}, \ldots, \tilde{\mathbf{T}}_N)$:

$$U_u(\tilde{\mathbf{T}}_u, \tilde{\mathbf{T}}_{-u}) \stackrel{\text{def}}{=} -\sum_{k\in\mathbf{S}_u} \text{tr}(\tilde{\mathbf{T}}_u^{(k)}\tilde{\mathbf{T}}_u^{(k)H}). \quad (6)$$

The transmitter of each link assigns power values over both the space and frequency dimensions, and configures its radiation pattern to maximize its own return. Formally, CR user u solves the following problem for its optimal precoders $\tilde{\mathbf{T}}_u$:

$$\begin{aligned} &\underset{\{\tilde{\mathbf{T}}_u^{(k)}, \forall k\in\mathbf{S}_u\}}{\text{maximize}} \quad U_u(\tilde{\mathbf{T}}_u, \tilde{\mathbf{T}}_{-u}) \\ &\text{s.t.} \quad \text{C1': } R_u \geq c_u \\ &\qquad \text{C2': } \text{tr}(\tilde{\mathbf{T}}_u^{(k)}\tilde{\mathbf{T}}_u^{(k)H}) \leq P_{\text{mask}}(u, f_k), \ \ \forall k \in \Psi_K \\ &\qquad \text{C3': } \sum_{k\in\Psi_K} \text{tr}(\tilde{\mathbf{T}}_u^{(k)}\tilde{\mathbf{T}}_u^{(k)H}) \leq P_{\max}. \end{aligned} \quad (7)$$

III. Existence and Uniqueness of the NE

Intuitively, three factors affect the existence of a NE of (7): Network (multi-user) interference, PU protection requirement (through power masks), and nodes' power budget. To deal with network interference, we first remove the power mask and power budget constraints (these constraints will be incorporated later) and have:

$$\begin{aligned} &\underset{\{\tilde{\mathbf{T}}_u^{(k)} \forall k\in\Psi_K\}}{\text{minimize}} \quad \sum_{k\in\mathbf{S}_u} \text{tr}(\tilde{\mathbf{T}}_u^{(k)}\mathbf{I}\tilde{\mathbf{T}}_u^{(k)H}) \\ &\text{s.t. C1' as in problem (7).} \end{aligned} \quad (8)$$

The precoding matrix $\tilde{\mathbf{T}}_u^{(k)}$ can be written as $\tilde{\mathbf{T}}_u^{(k)} = \mathbf{T}_u^{(k)} \times {\mathbf{P}_k^{(u)}}^{1/2}$ where $\mathbf{T}_u^{(k)}$ is an $M \times M$ matrix with unit-norm column vectors, specifying the directions to which the antenna array of node u points its beams. $\mathbf{P}_k^{(u)}$ is an $M \times M$ diagonal matrix whose entry (s, s) is the power allocated for sub-channel (s, k), $P_{s,k}^{(u)}$. Both $\mathbf{T}_u^{(k)}$ and $\mathbf{P}_k^{(u)}$ shape the antenna patterns.

At a NE, let $\boldsymbol{p}_u^{(k)} \stackrel{\text{def}}{=} (P_{1,k}^{(u)}, P_{2,k}^{(u)}, \ldots, P_{M,k}^{(u)})$ be a $1 \times M$ nonnegative vector, which denotes the power allocation vector of link u on its M antennas at frequency f_k (for $f_k \notin \mathbf{S}_u$, $\boldsymbol{p}_u^{(k)}$ is a zero vector). Let $\boldsymbol{p}_u \stackrel{\text{def}}{=} (\boldsymbol{p}_u^{(1)}, \boldsymbol{p}_u^{(2)}, \ldots, \boldsymbol{p}_u^{(K)})$ be a $1 \times MK$ vector, which denotes the power allocation on all antennas and frequencies of link u. Let $\boldsymbol{p} \stackrel{\text{def}}{=} (\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_N) \in \mathbb{R}_+^{NKM}$ denote the power allocation on all antennas and frequencies of all players.

We observe that the unit matrix $\mathbf{I}$ is positive definite, so the objective function in (8) is non-decreasing in every element of $\boldsymbol{p}_u$. In other words, at a NE of the game (if one exists), the inequality constraint C1' becomes equality. Otherwise, one can still lower the power consumption to achieve a smaller value for the objective function while meeting the rate demand. This fact defines a feasible set for $\boldsymbol{p}$, denoted by $\mathbb{Q}_{feasible}(\boldsymbol{c})$, corresponding to a given requested rate profile $\boldsymbol{c} \stackrel{\text{def}}{=} (c_1, c_2, \dots, c_N)$ at a NE. For a given rate profile $\boldsymbol{c}$, the game (8) has at least one bounded NE and only bounded NEs, if $\mathbb{Q}_{feasible}(\boldsymbol{c})$ is nonempty and bounded.

Theorem 1: Let $\mathbf{G}_k$ be defined in (11) and matrix $\mathbf{G}'_k$ be obtained from $\mathbf{G}_k$ by deleting rows $\mathbf{G}_k(u,:)$ and columns $\mathbf{G}_k(:,u)$ for all $\{u | k \notin \mathbf{S}_u\}$. If $\boldsymbol{G}'_k$ is a P-matrix[1] $\forall k \in \Psi_K$, then $\mathbb{Q}_{feasible}(\boldsymbol{c})$ contains at least one bounded vector $\boldsymbol{p} \in \mathbb{R}_+^{NKM}$ and only bounded vectors $\boldsymbol{p}$. In other words, the game (8) admits at least one bounded NE and only bounded NEs.

Proof: We first claim that $\mathbb{Q}_{feasible}(\boldsymbol{c})$ contains at least one bounded vector $\boldsymbol{p} \in \mathbb{R}_+^{NKM}$ or the existence of a bounded NE to the game (8):

Lemma 1: Given that $\boldsymbol{G}'_k$ is a P-matrix $\forall k \in \Psi_K$, then there exists at least one bounded vector $\boldsymbol{p} \in \mathbb{Q}_{feasible}(\boldsymbol{c}) \in \mathbb{R}_+^{NKM}$.

Proof: See Appendix I of [13]. □

The remaining task is to show that the game (8) admits only bounded NEs or $\mathbb{Q}_{feasible}(\boldsymbol{c})$ is bounded. To that end, we rely on the concept of asymptotic cone of a nonempty set in recession analysis [14]. For a nonempty set $\mathbb{Q} \in \mathbb{R}_+^N$, its *asymptotic cone*, denoted by $\mathbb{Q}_{asymp}$, consists of vectors $\boldsymbol{d} \in \mathbb{R}_+^N$, referred to as *limit directions*. Each limit direction vector $\boldsymbol{d}$ is defined through the existence of a sequence of vectors $\boldsymbol{p}_n \in \mathbb{Q}$ and a sequence of scalars ν_n tending to $+\infty$ such that [14]:

$$\lim_{n\to\infty} \frac{\boldsymbol{p}_n}{\nu_n} = \boldsymbol{d}. \tag{14}$$

The set $\mathbb{Q}$ is bounded if its asymptotic cone $\mathbb{Q}_{asymp}$ contains only the zero vector $\mathbf{0}$ [14]. Applying this to the set $\mathbb{Q}_{feasible}(\boldsymbol{c})$, the game (8) admits only bounded NEs if its asymptotic cone $\mathbb{Q}_{asymp}(\boldsymbol{c})$ contains only the zero vector. The asymptotic cone $\mathbb{Q}_{asymp}(\boldsymbol{c})$ is formally defined in (12).

Given that $\mathbb{Q}_{feasible}(\boldsymbol{c})$ has at least one bounded $\boldsymbol{p}$ (Lemma 1), it is clear that the vector zero $\mathbf{0}$ belongs to its asymptotic cone $\mathbb{Q}_{asymp}(\boldsymbol{c})$ (by the definition of limit directions). We now construct a set $\mathbb{Q}(\boldsymbol{c})$ of which $\mathbb{Q}_{asymp}(\boldsymbol{c})$ is a subset and prove that $\mathbb{Q}(\boldsymbol{c}) = \{\mathbf{0}\}$ if $\boldsymbol{G}'_{\boldsymbol{k}}$ is a P-matrix $\forall k \in \Psi_K$.

Lemma 2: If $\boldsymbol{d} \in \mathbb{Q}_{asymp}(\boldsymbol{c})$ then $\boldsymbol{d}$ belongs to $\mathbb{Q}(\boldsymbol{c})$, defined in (13).

Proof: See Appendix II in [13]. □

Assuming that there exists at least one $\boldsymbol{d} \neq \mathbf{0}$ and that $\boldsymbol{d} \in \mathbb{Q}(\boldsymbol{c})$, then $\forall u \in \Phi_N$ and $\mathbf{S}_u \supset k$:

$$\log\left(1 + \frac{\text{tr}(\tilde{\mathbf{T}}_u^{(k)H}\tilde{\mathbf{T}}_u^{(k)})|\mathbf{H}_{u,u}^{(k)H}\mathbf{H}_{u,u}^{(k)}|^{\frac{1}{M}}}{\sum\limits_{\{j|k\in\mathbf{S}_j\}} \frac{\text{tr}(\mathbf{H}_{u,j}^{(k)H}\mathbf{H}_{u,j}^{(k)})}{M}\text{tr}(\tilde{\mathbf{T}}_j^{(k)H}\tilde{\mathbf{T}}_j^{(k)})}\right) \leq c_u \tag{15a}$$

$$\text{tr}(\tilde{\mathbf{T}}_u^{(k)H}\tilde{\mathbf{T}}_u^{(k)})|\mathbf{H}_{u,u}^{(k)H}\mathbf{H}_{u,u}^{(k)}|^{\frac{1}{M}} - (2^{c_u}-1)\sum_{\{j|k\in\mathbf{S}_j\}} \text{tr}(\tilde{\mathbf{T}}_j^{(k)H}\tilde{\mathbf{T}}_j^{(k)})\frac{\text{tr}(\mathbf{H}_{u,j}^{(k)H}\mathbf{H}_{u,j}^{(k)})}{M} \leq 0 \tag{15b}$$

$$\mathbf{G}'_k \times [\text{tr}(\tilde{\mathbf{T}}_u^{(k)H}\tilde{\mathbf{T}}_u^{(k)}), \dots, \text{tr}(\tilde{\mathbf{T}}_j^{(k)H}\tilde{\mathbf{T}}_j^{(k)})]^T \leq \mathbf{0}. \tag{15c}$$

where $\mathbf{S}_u \supset k$ and $\mathbf{S}_j \supset k$.

As $\mathbf{G}'_k$ is a P-matrix for all $k \in \Psi_K$ and $[\text{tr}(\tilde{\mathbf{T}}_u^{(k)H}\tilde{\mathbf{T}}_u^{(k)}), \dots, \text{tr}(\tilde{\mathbf{T}}_j^{(k)H}\tilde{\mathbf{T}}_j^{(k)})]^T$ is a nonnegative vector, (15c) implies $\text{tr}(\tilde{\mathbf{T}}_u^{(k)H}\tilde{\mathbf{T}}_u^{(k)}) = 0 \ \forall u \in \Phi_N, \forall k \in \Psi_K$ [12] or $\boldsymbol{d} = \mathbf{0}$. This contradicts the above assumption. Hence, $\mathbb{Q}(\boldsymbol{c})$ and its subset $\mathbb{Q}_{asymp}(\boldsymbol{c})$ equal to $\{\mathbf{0}\}$. Theorem 1 is proved. □

We now give some intuitions behind Theorem 1. As the diagonal elements of $\mathbf{G}'_k$ are positive (under rich-scattering environment), then a sufficient condition for $\mathbf{G}'_k$ to be a P-matrix is $|\mathbf{G}'_k(u,u)| \geq \sum_{j\neq u} |\mathbf{G}'_k(u,j)|$ (i.e., row diagonally dominant) [12]. The following inequality guarantees that game (8) has at least one bounded NE and only bounded NEs:

$$\frac{M\det(\mathbf{H}_{u,u}^{(k)H}\mathbf{H}_{u,u}^{(k)})^{\frac{1}{M}}}{\sum\limits_{\{j|k\in\mathbf{S}_j\}} \text{tr}(\mathbf{H}_{u,j}^{(k)H}\mathbf{H}_{u,j}^{(k)})} \geq (2^{c_u}-1) \ \forall\ k\ ,\ \forall \text{ u}. \tag{16}$$

The nominator of the LHS in (16) represents the strength of the channel gain of link u on channel f_k, while its denominator describes the strength of cross-(interfering) channel gains from other links j, $j \neq u$, on the receiver of link u. First, for the game (8) to have at least one NE (at which the required powers of all links are bounded), the multi-user interference in each channel f_k should not be too strong. Second, the acceptable multi-user interference is explicitly quantified in (16), and is a function of the rate demand c_u of each link u. For higher rate demands, inequality (16) becomes stringent, meaning that lower multi-user interference is necessary. Hence, inequality (16) can be used as a criterion to reject or admit a newly requested transmission/rate. When links set their target rate too high that inequality (16) does not hold, a bounded NE may not exist. In this case, nodes keep increasing their transmit powers to meet their rate demands. Network interference becomes more severe and no link reaches its requested rate (interference-limited communications).

To better interpret inequality (16), recall that each element of channel gain matrices in (16) is the product of a complex Gaussian variable with zero mean and unit variance (in the $\bar{\mathbf{H}}_{u,u}^{(k)}$ matrix) and the distance-dependence attenuation factor: $\mathbf{H}_{u,u}^{(k)} = \frac{1}{\sqrt{d_{u,u}^n}}\bar{\mathbf{H}}_{u,u}^{(k)}$ where n is the free-space attenuation factor and $d_{u,u}$ is the transmission distance of link u. Inequality (16) can be rewritten as:

$$\frac{M\det(\bar{\mathbf{H}}_{u,u}^{(k)H}\bar{\mathbf{H}}_{u,u}^{(k)})^{\frac{1}{M}}}{\sum\limits_{\{j|k\in\mathbf{S}_j\}} \frac{d_{u,u}^n}{d_{u,j}^n}\text{tr}(\bar{\mathbf{H}}_{u,j}^{(k)H}\bar{\mathbf{H}}_{u,j}^{(k)})} \geq (2^{c_u}-1) \ \forall\ k\ ,\ \forall \text{ u}. \tag{17}$$

(17) holds if the distance between the transmitter and the receiver is small enough compared with distances between the receiver and its interferers, the channel gain matrix of link u is full-rank (this is often the case in a rich-scattering environment) and its requested rate is not too high. Given the existence of bounded NEs to the game in (8), we now incorporate the power mask and power budget constraints in the following theorem.

Theorem 2: The game (7) admits at least one bounded NE and only bounded NEs if $\boldsymbol{G}'_k$ is a P-matrix and the vector-inequality

[1] A matrix is a P-matrix if all of its principal minors are positive [12].

$$\mathbb{Q}_{feasible}(\boldsymbol{c}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{p} \in \mathbb{R}_+^{NKM} | R_u(\boldsymbol{p}) \stackrel{\text{def}}{=} \sum_{k \in \mathbf{S}_u} \log |\mathbf{I} + \tilde{\mathbf{T}}_u^{(k)H} \mathbf{H}_{u,u}^{(k)H} \mathbf{C}_u^{(k)^{-1}} \mathbf{H}_{u,u}^{(k)} \tilde{\mathbf{T}}_u^{(k)}| = c_u, \forall u \in \Phi_N \right\} \quad (10)$$

$$\mathbf{G}_k \stackrel{\text{def}}{=} \begin{bmatrix} |\mathbf{H}_{11}^{(k)H}\mathbf{H}_{1,1}^{(k)}|^{\frac{1}{M}} & -(2^{c_1}-1)\frac{\mathrm{tr}(\mathbf{H}_{1,2}^{(k)H}\mathbf{H}_{1,2}^{(k)})}{M} & \cdots & -(2^{c_1}-1)\frac{\mathrm{tr}(\mathbf{H}_{1,N}^{(k)H}\mathbf{H}_{1,N}^{(k)})}{M} \\ -(2^{c_2}-1)\frac{\mathrm{tr}(\mathbf{H}_{2,1}^{(k)H}\mathbf{H}_{2,1}^{(k)})}{M} & |\mathbf{H}_{2,2}^{(k)H}\mathbf{H}_{2,2}^{(k)}|^{\frac{1}{M}} & \cdots & -(2^{c_2}-1)\frac{\mathrm{tr}(\mathbf{H}_{2,N}^{(k)H}\mathbf{H}_{2,N}^{(k)})}{M} \\ \vdots & \vdots & \ddots & \vdots \\ -(2^{c_N}-1)\frac{\mathrm{tr}(\mathbf{H}_{N,1}^{(k)H}\mathbf{H}_{N,1}^{(k)})}{M} & -(2^{c_N}-1)\frac{\mathrm{tr}(\mathbf{H}_{N,2}^{(k)H}\mathbf{H}_{N,2}^{(k)})}{M} & \cdots & |\mathbf{H}_{N,N}^{(k)H}\mathbf{H}_{N,N}^{(k)}|^{\frac{1}{M}} \end{bmatrix} \quad (11)$$

$$\mathbb{Q}_{asymp}(\boldsymbol{c}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{d} \in \mathbb{R}_+^{NKM} | \exists \{\boldsymbol{p}_n\} \in \mathbb{Q}_{feasible}(\boldsymbol{c}) \text{ and } \{\nu_n\} \to +\infty \text{ so that } \lim_{n\to\infty} \frac{\boldsymbol{p}_n}{\nu_n} = \boldsymbol{d} \right\} \quad (12)$$

$$\mathbb{Q}(\boldsymbol{c}) \stackrel{\text{def}}{=} \left\{ \boldsymbol{d} \in \mathbb{R}_+^{NKM} | R'_u(\boldsymbol{d}) \stackrel{\text{def}}{=} \sum_{k \in \mathbf{S}_u} \log \left(1 + \frac{\mathrm{tr}(\tilde{\mathbf{T}}_u^{(k)H}\tilde{\mathbf{T}}_u^{(k)}) |\mathbf{H}_{u,u}^{(k)H}\mathbf{H}_{u,u}^{(k)}|^{\frac{1}{M}}}{\sum\limits_{\{j|k\in\mathbf{S}_j\}} \frac{\mathrm{tr}(\mathbf{H}_{u,j}^{(k)H}\mathbf{H}_{u,j}^{(k)})}{M} \mathrm{tr}(\tilde{\mathbf{T}}_j^{(k)H}\tilde{\mathbf{T}}_j^{(k)})} \right) \leq c_u, \forall u \in \Phi_N \right\} \quad (13)$$

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below holds element-by-element $\forall k \in \Psi_K$ and $\forall u \in \Phi_N$:

$$\mathbf{G}_k'^{-1} \times \begin{bmatrix} 2^{c_u}-1 \\ \vdots \\ 2^{c_j}-1 \end{bmatrix} \leq \begin{bmatrix} \frac{P_{\text{mask}}(f_k)}{1+I_{pu}(k)} \\ \vdots \\ \frac{P_{\text{mask}}(f_k)}{1+I_{pu}(k)} \end{bmatrix} \quad (18)$$

and

$$\sum_{k \in \mathbf{S}_u} (1 + I_{pu}(k)) \mathbf{G}_k'^{-1}(u,:) \times [2^{c_u}-1 \ldots 2^{c_j}-1]^T \leq P_{\max} \quad (19)$$

where each element of vector $[2^{c_u}-1 \ldots 2^{c_j}-1]^T$ corresponds to a link j that shares channel k with u (i.e., $k \in \mathbf{S}_j$ and $k \in \mathbf{S}_u$), $\mathbf{G}_k'^{-1}(u,:)$ is the uth row of the inverse of matrix $\mathbf{G}_k'$[2].

Proof: See proof of Theorem 2 in [13]. □

From (18), if PUs are more active on a given channel (higher $I_{pu}(k)$), the inequality becomes stricter. This means that CRs should reduce their transmission power on this channel to avoid interfering PUs. Moreover, as the inequality becomes tighter (smaller LHS of (18)) when PUs become more active, it is less likely for a NE to exist. Hence, besides the PU protection requirement, inequality (18) also shows the interference effect from PUs to CRs.

So far, we have derived conditions that capture the factors that affect the existence of a NE of the game (7). The conditions in Theorem 1 ensure that network interference is mild enough to support the requested rates. The conditions in the first inequality in Theorem 2 enforce that the requested rates are not too high to harm PUs reception given PUs' activities (indirectly captured by PUs' interference). The last inequality in Theorem 2 guarantees that rate demands are affordable given nodes' power budgets.

When the spectrum opportunities are homogeneous (i.e., $\mathbf{S}_u = \Psi_K, \forall u$), one can verify that by removing the resource and PUs protection constraints and setting the number of antenna to be one, the conditions in Theorem 1 reduce to the conditions derived for the NE existence in single-antenna (legacy) networks (in Theorem 5 of [8]). The authors of [8] proved that their sufficient conditions become necessary when $K = 1$ and $M = 1$, i.e., a single-channel SISO network (Proposition 11 of [8]). They also showed that for the case $K = 1$ and $M = 1$, their sufficient conditions are identical to those in [9]. Hence, though we cannot show that the sufficient conditions in Theorems 1 and 2 are also necessary in general cases, the following corollary gives a sense of how tight the conditions in Theorem 1 are.

Corollary 1: If $M = 1$ and spectrum opportunities are homogeneous, the conditions in Theorem 1 become the sufficient for the NE existence derived for the SISO network in [8]. Furthermore, If $K = 1$, then the sufficient conditions for a NE existence in Theorem 1 become necessary and identical to those in [9].

One may be curious about the relation between the NE existence and the fulfillment of rate demands. The following theorem shows that if the requested rates can be supported, then a NE must exist.

Theorem 3: If rate demands are supported, then the game (7) admits at least one NE.

Proof: Similar to the proof of Theorem 3 for the homogeneous spectrum sharing in [13]. □

Theorem 3 also points out that a NE does not exist only if the requested rates are not met. In such a case, players whose rates are not achieved have to reduce their demands (or even leave the game to reduce network interference and facilitate other links' transmissions), and then repeat game (7). Investigating this process would require a repeated game, which is left for a future work.

To analyze the uniqueness of the NE, we resort to variational inequalities theory, casting (7) as a variational inequalities (VI) problem [5].

Theorem 4: If game (7) has a NE, then this NE is unique.

Proof: See Appendix A. □

Theorem 4 indicates that (7) does not have multiple NEs. Hence, the NE existence condition of (7) is also the NE uniqueness condition (formally stated in Theorem 5 below).

Theorem 5: If the conditions in Theorem 2 hold, then there exists a unique NE of the game (7).

For the *best response*, each link needs to solve the individual utility optimization problem (7). (7) is a convex problem, which can be solved efficiently by standard solvers. One can also exploit the strong-duality to derive a low-complexity solution for (7). Due to space limit, we omit description of such a solution, which can be found in [13]. In [13], we can also prove the convergence to the game's unique NE.

IV. Numerical Results

We first numerically evaluate the conditions for the existence and uniqueness of a NE. To save the space, we consider a network of 2 links (link 1: node 1 to node 2; link 2: node 3 to node 4) and one channel. Each node is equipped with 2 antennas. Both links have

[2]Since $\mathbf{G}_k'$ is a P-matrix, it is invertible.

a rate demand of 3 bps/Hz. Channel gain matrices among the 4 nodes are in Section IV.B of [13] (where $H(:,:,i,j)$ is the channel gain matrix from node i to node j).

Conditions in Theorem 1:

$$\mathbf{G}_1' \stackrel{\text{def}}{=} \begin{bmatrix} |\mathbf{H}(:,:,2,1)\mathbf{H}(:,:,2,1)|^{\frac{1}{2}} & -(2^3-1)\frac{\text{tr}(\mathbf{H}(:,:,2,3)\mathbf{H}(:,:,2,3))}{2} \\ -(2^3-1)\frac{\text{tr}(\mathbf{H}(:,:,4,1)\mathbf{H}(:,:4,1))}{2} & |\mathbf{H}(:,:,4,3)\mathbf{H}(:,:4,3)|^{\frac{1}{2}} \end{bmatrix} = \begin{bmatrix} 0.0808 & -0.0241 \\ -0.0252 & 0.0748 \end{bmatrix} \quad (20)$$

The above $\mathbf{G}_1'$ is a P-matrix as it meets the sufficient conditions in (16). Hence, Theorem 1 holds.

Now, we check conditions in Theorem 2.

Conditions in Theorem 2 to protect PUs:

$$\mathbf{G}_1'^{-1} = \begin{bmatrix} 13.7588 & 4.4330 \\ 4.6353 & 14.8624 \end{bmatrix} \quad (21)$$

The inequality (18) to protect PUs is:

$$\mathbf{G}_1'^{-1} \times \begin{bmatrix} 2^3-1 \\ 2^3-1 \end{bmatrix} = \begin{bmatrix} 127.34 \\ 136.48 \end{bmatrix} \leq \begin{bmatrix} \frac{P_{\text{mask}}(f_1)}{1+I_{pu}(1)} \\ \frac{P_{\text{mask}}(f_1)}{1+I_{pu}(1)} \end{bmatrix} \quad (22)$$

The inequality (22) holds if the power mask is 136.48 times greater than $(1+I_{pu}(1))$ (note that $1+I_{pu}(1)$ is the total floor noise (normalized to 1) and the PUs' interference on channel 1, $I_{pu}(1)$). This is the case if PUs' interference is not too strong. If cognitive radios obtain temporarily idle ("white") channels from spectrum databases, then there is no active PUs (i.e., $I_{pu}(1)=0$). In this case, inequality (22) holds easily.

Conditions in Theorem 2 regarding the total power budget constraints:

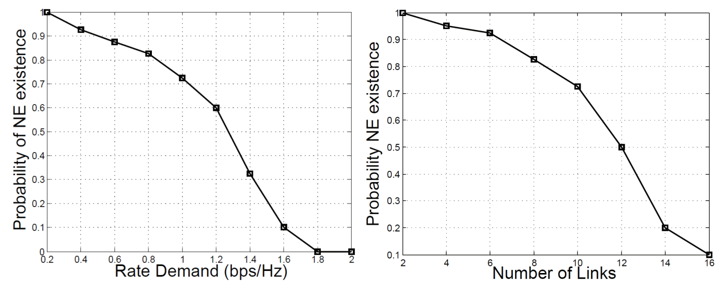
The LHS of (19) for the two links reduce to scalars in the considered example (as $K=1$) is:

$$\begin{aligned} &(1+I_{pu}(1))\times[13.7588 \quad 4.4330]\times[2^3-1 \quad 2^3-1]^T \\ &=127.34(1+I_{pu}(1)) \\ &(1+I_{pu}(1))\times[4.6353 \quad 14.8624]\times[2^3-1 \quad 2^3-1]^T \\ &=136.48(1+I_{pu}(1)) \end{aligned} \quad (23)$$

The second inequality (regarding the total power budget constraint) can also be met if the power budgets of link 1 and link 2 are greater than $127.34(1+I_{pu}(1))$ and $136.48(1+I_{pu}(1))$. Similar to the inequality (22), these conditions can also be met easily in practice. Note that as our example has only one channel (for simplicity), then the conditions regarding the power budget constraints are similar to that for the power mask constraints.

We now simulate a CMIMO network of N links (i.e., $2N$ nodes) which are randomly placed in a square area of length 100 meters. Each node has 4 antennas. The simulation results are averaged over 40 runs. There are 10 channels with bandwidth of 16 MHz. Due to spectrum heterogeneity, we assume that channels $i+1$, $i+2$ are not available for link i, if $i \leq 8$. Otherwise, channels $i-7$, $i-8$ are not available for link i. We set $P_{\max} = 1000$ mW and the power mask $P_{\text{mask}} = 0.5P_{\max}$ for all channels. The channel fading is flat with free-space attenuation factor of 2. The spreading angles of the signal at the receive antennas vary from $-\pi/5$ to $\pi/5$. The close-in distance is 1 m. The thermal floor noise is -174 dBm/Hz. The PUs interference on all channels is -100 dBm/Hz. We also assume that links have identical rate demands.

For a given simulation run, there is a probability that the conditions in Theorem 2 hold and the game converges to a unique NE. Fig. 1(a) depicts the probability (percentage of runs) that the game converges to a NE (a NE exists) versus the rate profile when 10 links are active and 10 channels are used. As the rate demand increases, the probability that a NE exists decreases. This is because the conditions in Theorem 2 become more stringent. Fig. 1(b) depicts the probability that a NE exists versus N when the rate demand is 1 bps/Hz. As N increases, the network/multi-user interference becomes more severe, it is unlikely that the conditions in Theorem 2 are met. Thus, the probability of a NE existence decreases.



(a) (b)

Fig. 1. (a) Probability of NE existence vs. rate demands, (b) Probability of NE existence vs. number of links.

Fig. 2 depicts the total power consumption for a network of 10 links with a rate demand of 1 bps/Hz. The game converges after a bout 13 iterations under Jacobi updates. Though we only prove the convergence under Gauss-Seidel update [13], simulations show that the game also converges under Jacobi and asynchronous updates. Fig. 3 shows the averaged number of iterations before reaching the NE under both synchronous (Jacobi and Gauss-Seidel) and asynchronous updating methods. For asynchronous updating, we allow odd-numbered links skip their updates every other iteration and even-numbered links skip their updates once every 3 iterations. As we can see, the game still converges to the NE under asynchronous update although its speed is slower than synchronous updates. When all players update their strategies simultaneously (Jacobi), the game converges faster. The difference in convergence speed of the Jacobi and Gauss-Seidel updates becomes more significant with the increase in the number of players.

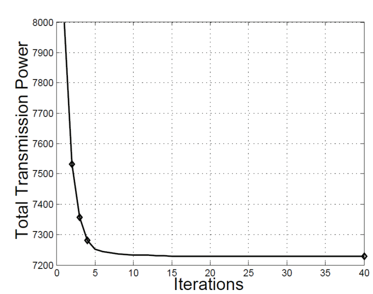


Fig. 2. Total network power consumption vs. iterations.

V. Conclusions

We derived sufficient conditions under which a cognitive MIMO network with heterogeneous spectrum sharing can support a given set of rate demands. By formulating the problem as a noncooperative game, using variational inequalities theory, and recession analysis, we derived sufficient conditions for the existence and uniqueness of the NE of the game. These conditions capture the interference from PUs, network interference of CMIMO, power

$$\text{vec}(\tilde{\mathbf{T}}_u) \overset{\text{def}}{=} \left[\Re[vec(\tilde{\mathbf{T}}_u^{(1)})]^T, ..., \Re[vec(\tilde{\mathbf{T}}_u^{(K)})]^T, \Im[vec(\tilde{\mathbf{T}}_u^{(1)})]^T, ..., \Im[vec(\tilde{\mathbf{T}}_i^{(K)})]^T \right] \leftrightarrow \boldsymbol{x} \in \mathbb{R}^{2KMM} \tag{24}$$

$$\nabla(.) \overset{\text{def}}{=} \left[\Re[vec(\frac{\partial(.)}{\partial \tilde{\mathbf{T}}_u^{(1)*}})]^T, ..., \Re[vec(\frac{\partial(.)}{\partial \tilde{\mathbf{T}}_u^{(K)*}})]^T, \Im[vec(\frac{\partial(.)}{\partial \tilde{\mathbf{T}}_u^{(1)*}})]^T, ..., \Im[vec(\frac{\partial(.)}{\partial \tilde{\mathbf{T}}_u^{(K)*}})]^T \right] \tag{25}$$

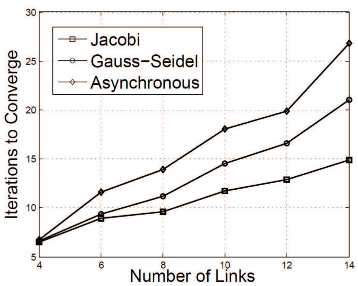


Fig. 3. Convergence speed vs. number of CR links.

budgets of CMIMO nodes. Using these conditions, a node can instantly decide if its requested rate can be supported.

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APPENDIX A
PROOF OF THEOREM 4

We start by introducing a VI problem.

Definition of a VI problem: [5] Given a subset $\mathbb{K}$ of the Euclidean n-dimensional space $\mathbb{R}^n$ and a mapping $F : \mathbb{K} \to \mathbb{R}^n$, the VI problem VI$(\mathbb{K}, \mathbb{R}^n)$ is to find a vector $\boldsymbol{x}^{\boldsymbol{opt}} \in \mathbb{K}$ so that:

$$(\boldsymbol{x} - \boldsymbol{x}^{\boldsymbol{opt}})^T F(\boldsymbol{x}^{\boldsymbol{opt}}) \geq 0, \quad \forall \boldsymbol{x} \in \mathbb{K}. \tag{26}$$

In the following, we state the sufficient conditions [5] for the existence and uniqueness of a solution to the above VI problem when the set $\mathbb{K}$ has a Cartesian structure, i.e., $\mathbb{K} = \mathbb{K}_1 \times \mathbb{K}_2 \times \ldots \times \mathbb{K}_N$ (where $\mathbb{K}_u \in \mathbb{R}^{n_u}$ and $\sum_{u=1}^{N} n_u = n$).

Theorem 6: Given that the set $\mathbb{K}$ has a Cartesian structure, the above VI$(\mathbb{K}, \mathbb{R}^n)$ problem admits a unique solution $\boldsymbol{x}^{opt}$ if $\mathbb{K}_u$ is closed and convex and F is continuous uniformly-P function, i.e, there exists a positive constant α such that:

$$\max_{\{1 \leq u \leq N\}} (\boldsymbol{x}_u - \boldsymbol{x}'_u)^T (F(\boldsymbol{x}_u) - F(\boldsymbol{x}'_u)) \geq \alpha \|\boldsymbol{x}_u - \boldsymbol{x}'_u\|^2, \quad \forall \boldsymbol{x}_u, \boldsymbol{x}'_u \in \mathbb{K}_u. \tag{27}$$

As the set of precoding matrices of each player in the game (7) are complex matrices. To reformulate the game (7) as a VI problem, we use the isomorphism in equation (24) to map the complex matrix domain to the Euclidean domain, where $vec()$ is a matrix operator that stacks columns (from left to right) of an $m \times n$ matrix to form an $mn \times 1$ vector.

The gradient of a matrix function $(.)$ w.r.t $\tilde{\mathbf{T}}_u$ is given in (25). We are now ready to map the game (7) to a VI problem. If all conditions in Theorem 2 are met, the strategy set of each player u, denoted by $Q_u \in \mathbb{C}^{M \times KM}$, is nonempty. Additionally, it is also easy to verify that Q_u is convex and bounded. Hence, problem (7) is a convex problem. The following inequality features the necessary (and then also the sufficient) condition for a strategy $\tilde{\mathbf{T}}_u^{opt}$ to be the best response:

$$(\tilde{\mathbf{T}}_u - \tilde{\mathbf{T}}_u^{opt}) \bullet \nabla U_u(\tilde{\mathbf{T}}_u, \tilde{\mathbf{T}}_{-u}) \leq 0 \quad \forall \tilde{\mathbf{T}}_u \in Q_u \tag{28}$$

where $\mathbf{A} \bullet \mathbf{B} \overset{\text{def}}{=} \text{vec}(\mathbf{A})^T \text{vec}(\mathbf{B})$.

Lets define $Q \overset{\text{def}}{=} Q_1 \times \ldots Q_N$ and $F \overset{\text{def}}{=} F_1 \times \ldots \times F_N$ with $F_u \overset{\text{def}}{=} -\nabla U_u(\tilde{\mathbf{T}}_u, \tilde{\mathbf{T}}_{-u})$. By comparing (28) with the above definition of a VI problem, the strategy set $\tilde{\mathbf{T}}^{opt} \overset{\text{def}}{=} [\tilde{\mathbf{T}}_1^{opt} \times \ldots \times \tilde{\mathbf{T}}_N^{opt}]$ is a NE of the game (7) if and only if $\tilde{\mathbf{T}}^{opt}$ is a solution of the VI(Q, F) problem. Therefore we can rely on VI theory to analyze the game (7).

Let $\tilde{\mathbf{T}} \overset{\text{def}}{=} [\tilde{\mathbf{T}}_1 \times \ldots \times \tilde{\mathbf{T}}_N]$ and $\tilde{\mathbf{T}}' \overset{\text{def}}{=} [\tilde{\mathbf{T}}'_1 \times \ldots \times \tilde{\mathbf{T}}'_N]$ be two different strategy set of the strategic space Q of the game (7), then:

$$\begin{aligned} F(\tilde{\mathbf{T}}'_u) &= -\nabla U_u(\tilde{\mathbf{T}}'_u, \tilde{\mathbf{T}}'_{-u}) = \tilde{\mathbf{T}}'_u \\ F(\tilde{\mathbf{T}}_u) &= -\nabla U_u(\tilde{\mathbf{T}}_u, \tilde{\mathbf{T}}_{-u}) = \tilde{\mathbf{T}}_u. \end{aligned} \tag{29}$$

Consequently, we have:

$$\text{vec}(\tilde{\mathbf{T}}_u - \tilde{\mathbf{T}}'_u)^T \text{vec}(F(\tilde{\mathbf{T}}_u) - F(\tilde{\mathbf{T}}'_u)) = 1||\text{vec}((\tilde{\mathbf{T}}_u - \tilde{\mathbf{T}}'_u))||^2. \tag{30}$$

The above inequality exactly meets the condition (27) so that the mapping F is a continuous uniformly-P function. Moreover, Q has a Cartesian structure. Hence, the VI(Q, F) problem has a unique NE, so does the game (7). $\square$